DR. MATHIAS BRAUN & WENHAO ZHAO

Exercises for 'Functional Analysis 2' [MATH-404]

(10/03/2025)

Ex 4.1 (Linear maps on L^p with $0 < p < 1$)
Let $p \in (0,1)$; let L^p denote the space of Lebesgue measurable functions on $\mathbb{R}$ for which

$$\rho(f) = \int_{\mathbb{R}} |f(x)|^p \, dx < +\infty,$$

endowed with the topology induced by the metric $d(f,g) = \rho(f-g)$.

a) Show that the only convex and open subsets of L^p are $\emptyset$ and L^p itself.
Hint: Since L^p is a TVS, wlog the open set contains the origin. Given $r > 0$ and $f \in L^p$, write $f = \sum_{i=0}^n \lambda_i g_i$ with $\lambda_i \in (0,1)$ and $\sum_i \lambda_i = 1$ and functions $g_i = \lambda_i^{-1} f \chi_{I_i}$, where the intervals I_i form a partition of $\mathbb{R}$ such that $\rho(g_i) < r$ for each i.

b) Let $T\colon L^p \to X$, where X is a LCTVS, be a continuous linear mapping. Prove that

$$Tf = 0 \quad \text{for all } f \in L^p.$$

c) Deduce that $(L^p)' = \{0\}$.

Solution 4.1 :
a) Let $U \subset L^p$ be convex, open and non-empty. We must show $U = L^p$. Since the metric is translation invariant and convexity is a translation-invariant property of sets, we may assume $0 \in U$. Thus since the topology is generated by the metric, there exists $r > 0$ such that $B_r = \{f : \ \rho(f) < r\} \subset U$.
It suffices to prove that for any $f \in L^p$, $f \in \mathrm{co}(B_r)$, the convex hull of B_r ; namely $f = \sum_{i=1}^n \lambda_i f_i$ for some functions $f_1, ..., f_n \in B_r$ and some $\lambda_1, ..., \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$. Since we know nothing about f, it makes sense to start by looking for such a decomposition with $\lambda_i = 1/n$ for all i. Recall in sheet 2 we were able to show the unit ball was not convex by considering convex combinations of functions with disjoint supports, taking this as our inspiration we seek to choose the f_i all with disjoint supports. In particular, we will prove that we can decompose $\mathbb{R}$ into n subintervals $I_1, ..., I_n$ with disjoint interiors such that

$$\bigcup_{j=1}^n I_j = \mathbb{R} \qquad \text{and} \qquad \int_{I_i} |f(x)|^p dx \le \frac{\rho(f)}{n} \tag{1}$$

We can then set $f_i = nf \cdot \mathbb{1}_{I_i}$ (here $\mathbb{1}_{I_i}(x)$ is the cut-off function equal to 1 if $x \in I_i$ and 0 if $x \notin I_i$) for which

$$\rho(f_i) = n^p \int_{I_i} |f(x)|^p dx \le n^{p-1} \rho(f)$$

for each $i = 1, ..., n$. Choosing $n > \left(\frac{\rho(f)}{r}\right)^{\frac{1}{1-p}}$ sufficiently large (using $p < 1$) each $f_i \in B_r$ and so

$$f = \sum_{i=1}^{n} \frac{1}{n} f_i \in \mathrm{co}(B_r)$$

and we are done.
We will now prove (1). Consider the function

$$F(y) = \int_{-\infty}^{y} |f(x)|^p dx,$$

This is continuous (e.g. by the monotone convergence theorem) and increasing with $\lim_{y\to-\infty} F(y) = 0$ and $\lim_{y\to\infty} F(y) = \rho(f)$. Thus, for each $z_i = \frac{i}{n}\rho(f)$, $i = 1, \ldots, n-1$, we can find y_i such that $F(y_i) = z_i$. This gives us a partition $-\infty =: y_0 < y_1 < \cdots < y_{n-1} < y_n := \infty$ such that

$$\int_{y_{i-1}}^{y_i} |f(x)|^p \, dx = F(y_i) - F(y_{i-1}) = \rho(f)/n.$$

(with appropriate adjustments at y_0 at y_n).

b) Let U be a neighbourhood of 0 in X. Since X is a LCTVS, U contains a convex open neighbourhood U' of 0. By continuity, $T^{-1}U'$ is open, and by linearity it contains 0 and is convex (check that if $Tf, Tg \in U'$ then $T(tf + (1-t)g) \in U'$ for all $t \in [0,1]$). Since $T^{-1}U'$ is non-empty we deduce from part (a) that it is all of L^p, i.e. $Tf \in U' \subset U$ for all $f \in L^p$. Since U was an arbitrary neighbourhood of 0 and X is Haussdorf (as a LCTVS), this implies that $Tf = 0$ for all $f \in L^p$.
c) Immediate, since any $T \in (L^p)'$ is a continuous linear mapping between L^p and $\mathbb{R}$ (which in particular is a LCTVS).

Ex 4.2 (The spaces $C(\Omega)$)
Let Ω be a nonempty open subset of $\mathbb{R}^d$.

a) Show that there exists a sequence of compact subsets $(K_n)_{n\in\mathbb{N}}$ such that

$$K_n \subset \mathrm{int}(K_{n+1}) \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} \mathrm{int}(K_n) = \Omega, \tag{$\star$}$$

where $\mathrm{int}(K)$ denotes the **interior** of a set K, i.e. the largest open set contained in K.
Hint: If $\Omega \neq \mathbb{R}^d$, work with the distance function to the closed set $\mathbb{R}^d \setminus \Omega$.

Let $C(\Omega)$ be the vector space of all continuous $f\colon \Omega \to \mathbb{R}$ and consider the family of seminorms

$$p_n(f) := \max_{x\in K_n} |f(x)|,$$

where $(K_n)_{n\in\mathbb{N}}$ is any sequence of compact sets satisfying ($\star$).

b) Show that $C(\Omega)$ with this family of seminorms is a LCTVS whose topology does not depend on the choice of a sequence $(K_n)_{n\in\mathbb{N}}$. Give an example of a translation invariant metric on $C(\Omega)$ and demonstrate that $C(\Omega)$ is a Fréchet space. Is it normable ?
c) Give an example of a bounded and closed set $E \subset C(\Omega)$ which is not compact.

Solution 4.2 :
a) We can for example put

$$K_n = \left\{x \in \Omega : \ |x| \leq n \text{ and } \mathrm{dist}(x, \mathbb{R}^d \setminus \Omega) \geq 1/n\right\},$$

with the condition $\text{dist}(x, \mathbb{R}^d \setminus \Omega) \geq 1/n$ left out if $\Omega = \mathbb{R}^d$. The interior of K_n is obtained by the same formula but with $\leq$ and $\geq$ replaced by $<$ and $>$, respectively.

b) To show that the seminorms $\{p_n\}_{n\in\mathbb{N}}$ induce a locally convex topology, assume that $p_n(f) = 0$ for all n. Then $f \equiv 0$ on every K_n, and from $(\star)$ it follows that $f \equiv 0$ on Ω. Moreover, the sets

$$B_n = \left\{ f \in C(\Omega) : \; p_n(f) < \frac{1}{n} \right\}$$

form a convex local neighborhood basis of the origin for $C(\Omega)$.

Now consider another sequence $(\tilde{K}_m)_{m\in\mathbb{N}}$ of compact sets that satisfy $(\star)$ and let $\tilde{p}_m$ denote the associated seminorms. Fix $n \in \mathbb{N}$ and let $B_n = \{f : \; p_n(f) < 1/n\}$ be a neighborhood basis of 0. Since $K_n \subset \Omega = \bigcup_m \text{int}(\tilde{K})_m$ and K_n is compact, there exist m_0 such that $K_n \subset \tilde{K}_{m_0}$. As a consequence $p_n \leq \tilde{p}_{m_0}$ and $\tilde{B}_{m_0,n} = \{f : \tilde{p}_{m_0}(f) < 1/n\}$ is an open set in the topology induced by $(\tilde{p}_m)_{m\in\mathbb{N}}$ such that $\tilde{B}_{m_0,n} \subset B_n$. We can demonstrate in the same way that for every neighborhood basis $\tilde{B}_m = \{f : \tilde{p}_m(f) < 1/m\}$ of 0 there exist $B_{n_0,m}$ such that $B_{n_0,m} \subset \tilde{B}_m$.

Let $\{f_k\}_k$ be a Cauchy sequence in $C(\Omega)$. Then $p_n(f_j - f_k) \to 0$ for every n, as $j, k \to +\infty$. Thus, for each n the sequence $\{f_k\}_k$ converges uniformly on K_n to a continuous function f_n defined on K_n (here we employ the completeness of the spaces $C(K_n)$ with the max norm). Since $(K_n)_{n\in\mathbb{N}}$ forms an increasing family of subsets of Ω, we get from uniqueness of the limits that $f_n = f_{n+1}$ on each K_n. Therefore we obtain a continuous function f on $\Omega = \bigcup_n \text{int}(K_n)$ such that $p_n(f_k - f) \to 0$ for each n as $k \to +\infty$. This implies that $C(\Omega)$ is sequentially complete (using question 3.1(a) from Exercise Sheet 3). Because it can be metrized with for example

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(f-g)}{1 + p_n(f-g)}$$

it is a Fréchet space.

Recall that $E \subset C(\Omega)$ is bounded iff there are numbers $M_n > 0$ such that $p_n(f) \leq M_n$ for every $f \in E$. Consider the neighborhood basis B_n of the origin. For any $f \in B_n$, we only know that $p_n(f) < 1/n$, so we can always construct a sequence $f_k \in B_n$ of continuous functions which grow unboundedly on $K_{n+1} \setminus K_n$; this can be done for example using question 3.4(a) of Exercise Sheet 3 to construct a function which is supported on $\overline{B_r(x_0)}$ for some $\overline{B_r(x_0)} \subset \Omega \setminus K_n$, see part (c) below for an alternative explicit construction. Then $p_{n+1}(f_k) \to +\infty$ as $k \to +\infty$, so B_n is not bounded. Kolmogorov's criterion implies that $C(\Omega)$ is not normable.

c) Fix any $n > 1$. Then $K_n \subset \Omega$ is compact and $\text{int}(K_n) \supset K_{n-1}$ is non-empty. Since $\text{int}(K_n) \subset \Omega$ is an open set, we can find a compact exhaustion $(\widetilde{K}_k)_k$ of $\text{int}(K_n)$ as in part (a). Fix $x_k \in \text{int}(\widetilde{K}_{k+1}) \setminus \widetilde{K}_k$ and let $l_k = \min\{\text{dist}(x_k, \widetilde{K}_k), \text{dist}(x_k, \partial\widetilde{K}_{k+1})\} > 0$, where $\partial\widetilde{K}_{k+1}$ is the boundary of $\widetilde{K}_{k+1}$. Consider the functions

$$f_k(x) = \max\left\{1 - |x - x_k|/l_k, 0\right\}, \quad x \in \Omega.$$

Then each f_k is continuous with support in $\widetilde{K}_{k+1} \setminus \text{int}\widetilde{K}_k \subset \text{int}K_j$ and $\max_\Omega |f_k| = 1$ for all k ; thus $E := \{f_k : k \in \mathbb{N}\}$ is bounded in $C(\Omega)$. Since the supports of the f_k are disjoint, for any $k \neq k'$, $p_m(f_k - f_{k'}) = 1$ for all $m > n$. Since any convergent sequence is Cauchy [1] this implies that any convergent sequence in E is eventually constant, so E is closed. In the same way, the sequence $(f_k)_k \subset E$ has no convergent subsequence, so E is not compact.

Ex 4.3 (Continuous functionals)

For each LCTVS X and a linear functional Λ on X, show that Λ is continuous :

1. Exercise ! Use continuity of addition as in Exercise 3.1(c).

a) $X = C(\Omega)$, $\Lambda_{x_0}(f) = f(x_0)$, where $x_0 \in \Omega$;

b) $X = C(\Omega)$, $\Lambda_g(f) = \int_\Omega f(x)g(x)\,dx$, where g is continuous with compact support in Ω;

c) $X = \mathcal{D}_{[a,b]}$, $\Lambda_{x_0}^{(k)}(f) = f^{(k)}(x_0)$, where $k \in \mathbb{N}_0$ and $x_0 \in [a,b]$.

Solution 4.3 : We use the notation of Exercise 4.2. To check continuity, we use the last part of Proposition 1.26.

a) It is obvious that Λ_{x_0} is a linear functional on $C(\Omega)$. For continuity, let K_n be a compact subset of Ω such that $x_0 \in K_n$, see $(\star)$. Then $|\Lambda_{x_0}(f)| = |f(x_0)| \leq p_n(f)$.

b) The linearity of the integral gives the linearity of Λ_g. Let $K \subset \Omega$ be the support of g. For continuity, we once more invoke $(\star)$ to find n such that $K \subseteq K_n$. Then

$$|\Lambda_g(f)| \leq p_n(f) \cdot \int_K |g(x)|\,\mathrm{d}x.$$

c) The linearity of $\Lambda_{x_0}^{(k)}$ follows from the linearity of differentiation. Recall that the topology in $\mathcal{D}_{[a,b]}$ is induced by the seminorms $p_n(f) = \max\{|f^{(k)}(x)|, x \in \mathbb{R},\ k \leq n\}$. Thus, for every f, $|\Lambda_{x_0}^{(k)}(f)| \leq p_k(f)$.

Ex 4.4 (Locally compact Hausdorff-TVS are finite dimensional)

Let X be a Hausdorff topological vector space such that 0 has an open neighborhood U with $\overline{U}$ being compact. Show that X is finite dimensional. You may follow the guideline below :

a) Show that there exists $x_1, \ldots, x_n \in X$ such that $\overline{U} \subset \bigcup_{i=1}^n (x_i + \frac{1}{2}U)$.

b) Define $Y = \operatorname{span}(x_1, \ldots, x_n)$ and deduce that Y is closed. Moreover, show that $\frac{1}{2}U \subset Y + \frac{1}{4}U$.

c) Prove by induction that

$$U \subset \bigcap_{n=1}^{+\infty} (Y + 2^{-n}U).$$

d) Deduce that $U \subset Y$ and finally $X \subset Y$ to conclude.